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# A Characterization of Commutative Rings in which every Semi Co-Hopfian Module is Artinian

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## Abstract

Let $R$ be an associative ring with unit $1 \neq 0$, we call an unital left $R$-module $M$ semi co-Hopfian (resp semi Hopfian ) if any injective (resp. surjective) endomorphism of $M$ has a direct summand image (resp kernel). Starting from that every artinian module is co-Hopfian and so semi co-Hopfian, we showad in this paper that the class of ring on which every semi co-Hopfian module is artinian coincide with the class of artinian principal ideal rings when the v.p is satisfied. Moreover, some properties of this class of rings are given.

Keywords: Semi co-Hopfian, Artinian, Vanaja property, $SCHA$-ring.

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## 1. Introduction

The study of classes of rings (modules) by properties of their endomorphisms is a classical research subject since 1960s. In 1986, Hiremath [3] introduced the concept of Hopfian modules and rings. Later, the dual concept cohopfian modules and rings were given. Hopfian and co-Hopfian modules (rings) have been investigated by several authors. In 2008, P. AYDOGDU and A. C. OZCAN introduced and investigated the semi Hopfian and semi co-Hopfian modules.

Recall that a module $M$ is called co-Hopfian (resp. Hopfian) if any injective (resp. surjective) endomorphism of $M$ is an automorphism. Note that any artinian (resp noetherian) module is co-Hopfian (resp. Hopfian).

A module $M$ is called semi co-Hopfian (resp. semi Hopfian) if any injective (resp. surjective) endomorphism of $M$ has a direct summand image (resp. kernel). In other words, any injective (resp. surjective) endomorphism of $M$ splits.

Clearly, any co-Hopfian (resp. Hopfian) module is semi co-Hopfian (resp semi Hopfian). The converse is not true in general, for example let $\mathbb{Z}_M = \mathbb{Q}^{(\mathbb{N})}$. By [2] $M$ is semi co-Hopfian but not co-Hopfian.

In [2] $M$ is semi co-Hopfian if and only if any submodule $N$ of $M$ which is isomorphic to $M$ is a direct summand of $M$, therefore, the concept of semi co-Hopfian module is a generalization of co-Hopfian module.

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The aim of this paper is to characterize the rings $R$, on which any semi co-Hopfian module is artinian. These rings are named $SCHA$-rings.
Throughout this paper, $R$ denotes an associative ring with identity $1 \neq 0$, and modules $M$ are unitary left $R$-modules. The property $Hom(M_i, M_j) = Hom(M_j, M_i) = 0$ whenever $i \neq j$ for a family $\{M_i\}$ of $R$-modules, is named Vanaja property (briefly v.p).
Let $R$ be a ring. A module $M$ is noetherian (resp. artinian) if any ascending (resp. descending) chain of submodules of $M$ stabilizes. $M$ is strongly co-Hopfian (resp. strongly Hopfian) if for any endomorphism $f$ of $M$ the descending chain $Im(f) \supseteq Im(f^2) \supseteq \cdots$ (resp. ascending chain $Ker(f) \subseteq Ker(f^2) \subseteq \cdots$ ) stabilizes. $M$ is called Dedekind finite if $M = M \oplus N$ for some module $N$, $N = 0$. The socle of $M$ ($Soc(M)$) is defined to be the sum of the minimal nonzero submodules of $M$. A submodule of $M$ is essential if it has a non-trival intersection with every non-trivial submodule of $M$: that is, $E \cap L = 0$ implies $L = 0$ for a submodule $L$ of $M$. $M$ is finitely cogenerated if only if $Soc(M)$ is essential and finitely generated.

## 2. Preliminary results

Lemma 2.1. The following are equivalent for a module $M$.

1. $M$ is co-Hopfian
2. $M$ is Dedekind finite and semi co-Hopfian
3. $M$ is weakly co-Hopfian and semi co-Hopfian.

Proof. (3) $\Leftrightarrow$ (1) $\Rightarrow$ (2) obvious
(2) $\Rightarrow$ (1) Let $f$ be an injective endomorphism of $M$. Then $f(M) \oplus K$ for $K \leqslant M$. Define a homomorphism $\varphi : M \oplus K \longrightarrow M$ by $\varphi(m, k) = f(m) + k$. Then $\varphi$ is an isomorphism. Since $M$ is Dedekind finite, $K = 0$. Hence $f(M) = M$ and so $f$ is an isomorphism. □

Lemma 2.2. Any direct summand of semi co-Hopfian modules is semi co-Hopfian.

Proof. Let $N$ be a direct summand of $M$ and $f : N \longrightarrow N$ a monomorphism. Write $M = N \oplus N'$. Then $g : M \longrightarrow M$, $g(n + n') = f(n) + n'$ where $n \in N$, $n' \in N'$, is a monomorphism. Since $Im(g) = Im(f) \oplus N'$ is a direct summand of $M$, we get that $Im(f)$ is a direct summand of $N$. □

Lemma 2.3. (Theorem .0.1. of [4] )

1. Let $M$ be a co-Hopfian (Hopfian) module. If $M$ decomposes as a direct sum of a family $\{M_i\}$ of nontrivial $R$-modules, then each $M_i$ is co-Hopfian (Hopfian).
2. Let $\{M_i\}$ be a family of family of nontrivial $R$-modules. We suppose that v.p is satisfied. If each $M_i$ is co-Hopfian (Hopfian), then so is $M$.

Lemma 2.4. (Proposition 10.18 of [1])
For each ring $R$, the following statements are equivalent:

1. $R$ is left artinian;
2. Every finitely generated left $R$-module is finitely cogenerated.

Remark 2.5. :

1. Every cyclic module is finitely generated.
2. Any co-Hopfian module is semi co-Hopfian. But the converse is not true in general. For example, let ${}_{\mathbb{Z}}M = \mathbb{Q}^{(\mathbb{N})}$. Since $M$ is quasi-injective, it is semi co-Hopfian. But $M \cong M \oplus \mathbb{Q}$ is not Dedekind finite, hence not co-Hopfian.

Lemma 2.6. Every finitely cogenerated module $M$ is Dedekind finite.

Proof. At beginning recall that if $\mathsf{Soc}(M)$ is essential and Dedekind finite, then $M$ is Dedekind finite. $M$ finitely cogenerated implies $\mathsf{Soc}(M)$ is essential. Now let's prove that $\mathsf{Soc}(M)$ is Dedekind finite. By definition $\mathsf{Soc}(M)$ is a direct sum of all simple submodules of $M$, hence $\mathsf{Soc}(M)$ is a semisimple submodule of $M$. We can write $\mathsf{Soc}(M) = \bigoplus_{i \in I} S_i$. Since $\mathsf{Soc}(M)$ is finitely generated then I is finite and so $\mathsf{Soc}(M)$ is of finite lenght. Therefore $\mathsf{Soc}(M)$ is Dedekind finite. □

An I-ring (S-ring) is a ring such that every co-Hopfian (Hopfian) module is artinian (noetherian).

Lemma 2.7. (Theorem 9 of [5])
For a commutative ring $R$, the following are equivalent:

1. $R$ is a artinian principal ideal ring;
2. $R$ an I-ring;
3. $R$ is S-ring.

Lemma 2.8. (Lemma 2 of [5] p.247)
Every integral domain S-ring is a field.

## 3. Aim results

Proposition 3.1. Let $R$ be a commutative ring. If $R$ is a $\mathsf{SCHA}$-ring, then $R$ is an I-ring.

Proof. Assume that $R$ is a $\mathsf{SCHA}$-ring. Let $M$ be a co-Hopfian module, then by second point of remark 2.5, $M$ is semi co-Hopfian. Therefore $M$ is artinian. □

Theorem 3.2. Let $R$ be a commutative ring. We suppose that v.p is satisfied. The following are equivalent:

1. $R$ is a artinian principal ideal ring;
2. $R$ is $\mathsf{SCHA}$-ring.

Proof. (2) ⇒ (1) Results from Lemma 2.6 and proposition 3.1.
(1) ⇒ (2) Let $M$ be a semi co-Hopfian module. $R$ principal ideal ring implies every $R$-module is a direct sum of cyclic modules. Let $M = \bigoplus M_i$. We can write $M = M_i \oplus (\oplus_{i \neq j}(M_j))$, in fact every $M_i$ is a direct summand of $M$. By lemma 2.2, every $M_i$ is semi co-Hopfian.($\star$)
Recall every $M_i$ is cyclic hence finitely generated. Since $R$ artinian, referring to lemma 2.4, for each $i$, $M_i$ is finitely cogenerated. It results by lemma 2.6 for each $i$ $M_i$ is Dedekind finite. ($\star\star$).
($\star$) and ($\star\star$) imply for each $i$ , $M_i$ is Dedekind finite and semi co-Hopfian. By lemma 2.1 for each $i$, $M_i$ is co-Hopfian. Referring to the second point of lemma 2.3 $M$ is co-Hopfian. Over artinian principal ideal ring, co-Hopfian and artinian modules coincide. Therefore $M$ is artinian.
Conclusion $R$ is a $\mathsf{SCHA}$-ring. □

Corollary 3.3. For a commutative ring $R$, if v.p is satisfied then the following are equivalent:

1. $R$ is an artinian principal ideal ring;
2. $R$ is an I-ring;
3. $R$ is a $\mathsf{SCHA}$-ring.
4. $R$ is a S-ring

Proof. Results from theorem 9 of [5] and theorem 3.2. □

Proposition 3.4. Let $R$ be a commutative $\mathsf{SCHA}$-ring. We suppose that v.p is satisfied. Then every prime ideal is maximal. Also, there are only finitely many prime ideals.

Proof. Let $P$ be a prime ideal of $R$, $R/P$ is a commutative integral domain. By referring successively corollary 3.3 and lemma 2.8, $R/P$ is a field. Therefore $P$ is maximal.
For the second statement we denote by $L$ the set of all prime ideals. Let $P \in L$, $R/P$ is a field so $R/P$ is simple. Furthemore if $P$ and $P' \in L$ with $P$
$nsubseteq P'$. Since v.p is satisfied then $Hom(R/P, R/P') = 0$. Every simple module is semi co-Hopfian but also Dedekind finite. By lemma 2.1 $R/P$ is co-Hopfian. If v.p is satisfied, direct sum of co-Hopfian modules is co-Hopfian. Therefore $M = \bigoplus_{P \in L} R/P$ is co-Hopfian. Over $SCHA$-ring every co-Hopfian module is artinian. Hence $L$ is finite. □

Lemma 3.5. (Theorem of [6]) Any injective endomorphism of a finitely generated $R$-module is an isomorphism if and only if every prime ideal of $R$ is maximal.
In other words all finitely generated $R$ module is co-Hopfian if and only if all prime ideals of $R$ are maximal.

Recall a ring $R$ is strongly-$\pi$-regular if every cyclic $R$-module is strongly co-Hopfian.

Proposition 3.6. Let $R$ be a commutative $SCHA$-ring. We suppose that v.p is satisfied. Then $R$ is strongly-$\pi$-regular.

Proof. Let $M$ be a cyclic $R$-module, then $M$ is finitely generated. By lemma 3.5 and proposition 3.4, $M$ is co-Hopfian. Since the class of $SCHA$-rings and the class of I-rings coincide if v.p is satisfied, then $M$ is artinian. Every artinian ring is strongly co-Hopfian, therefore $M$ is strongly co-Hopfian.
In conclusion $R$ is strongly-$\pi$-regular. □

Proposition 3.7. Let's suppose v.p is satisfied, A finite direct product $R = \prod R_i$ of $SCHA$-rings is a $SCHA$-ring if and only if every $R_i$ is a $SCHA$-ring.

Proof. First let's suppose $R = \prod R_i$ is a $SCHA$-ring and $M_i$ a semi co-Hopfian $R_i$-module. From this surjective homomorphism $P_i : \prod R_i \longrightarrow R_i$ every $R_i$-module has a structure of $R$-module. Hence $M_i$ is artinian.
Secondly, let's suppose every $R_i$ is a $SCHA$-ring. By v.p we can write $M = \oplus M_i = M_i \oplus (\oplus_{i \neq j}(M_j))$ with each $M_i$ is a $R_i$-module. $M$ semi co-Hopfian by lemma 2.2 every $M_i$ is semi co-Hopfian. Therefore every $M_i$ is artinian. A finite direct product of artinian rings is artinian. In conclusion $R = \prod R_i$ is a $SCHA$-ring. □

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